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Energy levels of the soliton–heavy-meson bound states

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Abstract

We investigate the bound states of heavy mesons with finite masses to a classical soliton solution in the Skyrme model. For a given model Lagrangian we solve the equations of motion *exactly* so that the heavy vector mesons are treated on the same footing as the heavy pseudoscalar mesons. All the energy levels of higher grand spin states as well as the ground state are given over a wide range of the heavy meson masses. We also examine the validity of the approximations used in the literatures. The recoil effect of finite mass soliton is naïvely estimated.

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I. INTRODUCTION

The bound state approach of Callan and Klebanov [1] has widened the applicability of the Skyrme model [2] up to heavy flavored baryons. In this approach, heavy baryons are described by bound states of the soliton of an $SU(2)$ chiral Lagrangian and the heavy mesons containing the corresponding heavy flavor. This picture originally introduced for the strange baryons was suggested to be applied for the heavy baryons containing one or more heavy quarks such as charm (c) and bottom (b) quarks by Rho, Riska, and Scoccola [3,4]. The results on the mass spectra [4] and magnetic moments [5] for the charmed baryons were found to be strikingly close to the quark model description.

Although qualitatively successful, when straightforwardly extended to the heavy flavors, the way of treating the heavy vector mesons in the traditional bound state approach [6] cannot be compatible with the heavy quark symmetry [7]. As far as the strange flavor is concerned, on the analogy of the case of ρ and π, the vector mesons K^* may be integrated out via *an ansatz* in favor of a combination of the background and the pseudoscalar meson field K. (See Sec. III for its detailed form.) However, such an approximation is valid only when the vector meson mass is much larger than that of the pseudoscalar meson. Furthermore, the ansatz suppresses the vector meson fields by a factor inversely proportional to the vector meson mass, while the heavy quark symmetry implies that the role of the vector mesons becomes as important as that of the pseudoscalar mesons in the heavy quark mass limit.

In the work of Jenkins, Manohar, and Wise [8] followed by a burst of publications [9–12], this problem has been neatly solved out. There, the bound state approach is applied to the heavy meson effective Lagrangian [13] that explicitly incorporates the heavy quark symmetry and the chiral symmetry on the same footing. The resulting baryon mass spectra show the correct hyperfine splittings consistent with the heavy quark symmetry. It emphasizes the essential role of the heavy vector mesons in the binding mechanism. This model is thoroughly studied in Ref. [14] for higher spin states and is further applied to the pentaquark exotic baryons [15].

However, these works are carried out in the limit of both the number of color N_c and the heavy quark mass m_Q going to infinity, where the soliton and the heavy mesons are infinitely heavy and so sit on the top of each other. This naïve picture would definitely yield larger binding energies to the heavy-meson–soliton system. Besides, degeneracies and parity doubling in the heavy baryon spectroscopy of Ref. [14] would be just an artifact originating from this assumption. In Ref. [16], the kinetic effects of finite heavy mesons have been estimated up to $1/m_Q$ order for the ground state. It is shown that the kinetic effects amounts to about 0.3 GeV in the charm sector, which is not small compared with the leading order binding energy of 0.8 GeV. Such a correction is expected to be more serious for the loosely bound pentaquark exotic states with a binding energy about 0.3 GeV.

In this paper, on our way of investigating the pentaquark exotic baryons and the heavy baryon excited states in a more realistic way [17], we generalize the work of Ref. [16] to obtain the heavy meson (and antiflavored heavy meson) bound states of higher grand spin and radially excited bound states. For a given model Lagrangian, we solve the equations of motion *exactly* without adopting any approximations. It enables us to investigate the bound states over a wide range of the heavy meson masses from the strangeness sector to the bottom sector, and allows us to examine the validity of the approximations used in the

previous works. We work simply *in the soliton-fixed frame*, neglecting any recoil effects due to the finite soliton mass. At the end, a naïve estimation of the soliton recoil effects is made by replacing the heavy meson masses by their reduced ones.

This paper is organized as follows. In the next section we briefly describe our model Lagrangian for completeness. Then, in Sec. III, we derive the equations of motion for the heavy mesons under the influence of the static potentials and analyze the symmetries of the equations. In Sec. IV we discuss the spherical solutions of the free Proca equations before we give our numerical results for the wave functions of the bound states in Sec. V where the energy levels of the soliton–heavy-meson system are presented and discussed. We summarize the detailed formulas for the grand spin eigenstates in Appendix A and the equations of motion for the radial functions in Appendix B.

II. MODEL LAGRANGIAN

We work with a simple model Lagrangian for a system of Goldstone bosons and the heavy mesons, which possesses the chiral symmetry explicitly and restores the heavy quark symmetry as the heavy meson mass goes to infinity.

As for the Goldstone bosons, we adopt the Skyrme model Lagrangian [2]

$$\mathcal{L}_M^{\rm SM} = \frac{f_\pi^2}{4}{\rm tr}(\partial_\mu U^\dagger \partial^\mu U) + \frac{1}{32e^2}{\rm tr}[U^\dagger\partial_\mu U, U^\dagger\partial_\nu U]^2, \tag{2.1a}$$

where f_π is the pion decay constant ($\approx$ 93 MeV empirically) and U is an $SU(2)$ matrix of the chiral field; viz.,

$$U = {\rm e}^{iM/f_\pi}, \tag{2.1b}$$

with M being a 2×2 matrix of the pion triplet

$$M = \boldsymbol{\tau}\cdot\boldsymbol{\pi} = \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}. \tag{2.1c}$$

Here, the chiral $SU(2)_L\times SU(2)_R$ symmetry is realized nonlinearly under the transformation of U:

$$U \longrightarrow LUR^\dagger, \tag{2.1d}$$

with $L\in SU(2)_L$ and $R\in SU(2)_R$. By the help of the Skyrme term with the Skyrme parameter e, the Lagrangian $\mathcal{L}_M^{\rm SM}$ supports a stable winding-number-1 soliton solution under the "hedgehog" configuration

$$U_0(\mathbf{r}) = \exp[i\boldsymbol{\tau}\cdot\hat{\mathbf{r}}F(r)], \tag{2.2}$$

where the profile of $F(r)$ is subject to the boundary condition, $F(0)=\pi$ and $F(r)\to 0$ as r becomes infinity. The solution carries a finite mass of order N_c and the winding number is interpreted as the baryon number. The pion decay constant and the Skyrme parameter

are fixed as f_π=64.5 MeV and e=5.45, respectively, so that the quantized soliton fits the nucleon and Δ masses [18].

Now, consider $j^\pi = 0^-$ and 1^- heavy mesons with quantum numbers of a heavy quark Q and a light antiquark $\bar{q}$. We take them as point-like objects described by the fields Φ and Φ^*_μ, respectively, which form $SU(2)$ anti-doublets. For example, if the heavy quark is charm flavored, they are the D-meson anti-doublets:

$$\Phi = (D^0, D^+) \quad \text{and} \quad \Phi^* = (D^{*0}, D^{*+}). \tag{2.3}$$

Their conventional free field Lagrangian density is given by

$$\mathcal{L}^{\text{free}}_\Phi = \partial_\mu \Phi \partial^\mu \Phi^\dagger - m^2_\Phi \Phi\Phi^\dagger - \tfrac{1}{2}\Phi^{*\mu\nu}\Phi^{*\dagger}_{\mu\nu} + m^2_{\Phi^*}\Phi^{*\mu}\Phi^{*\dagger}_\mu, \tag{2.4}$$

where $\Phi^*_{\mu\nu} = \partial_\mu \Phi^*_\nu - \partial_\nu \Phi^*_\mu$ is the field strength tensor of the heavy vector meson fields Φ^*_μ and m_Φ (m_{Φ^*}) is the mass of the heavy pseudoscalar (vector) mesons.

In order to construct a chirally invariant Lagrangian containing Φ, Φ^*_μ, and their couplings to the Goldstone bosons, we need to assign a transformation rule with respect to the chiral symmetry group $SU(2)_L \times SU(2)_R$ to the heavy meson fields. For this end, we introduce

$$\xi = U^{\frac{1}{2}}, \tag{2.5}$$

which transforms under the $SU(2)_L \times SU(2)_R$ as

$$\xi \to \xi' = L\xi\vartheta^\dagger = \vartheta\xi R^\dagger, \tag{2.6}$$

where ϑ is a local unitary matrix depending on L, R, and the Goldstone fields $M(x)$. Then, we can construct a vector field V_μ and an axial vector field A_μ as

$$\begin{aligned} V_\mu &= \tfrac{1}{2}(\xi^\dagger\partial_\mu\xi + \xi\partial_\mu\xi^\dagger), \\ A_\mu &= \tfrac{i}{2}(\xi^\dagger\partial_\mu\xi - \xi\partial_\mu\xi^\dagger). \end{aligned} \tag{2.7}$$

The vector field V_μ behaves as a gauge field under the local chiral transformation while the axial vector field transforms covariantly; viz.,

$$\begin{aligned} V_\mu &\to V'_\mu = \vartheta V_\mu \vartheta^\dagger + \vartheta\partial_\mu\vartheta^\dagger, \\ A_\mu &\to A'_\mu = \vartheta A_\mu \vartheta^\dagger. \end{aligned} \tag{2.8}$$

In terms of ϑ, the chiral transformations of Φ and Φ^* fields are expressed as

$$\Phi \to \Phi' = \Phi\vartheta^\dagger \quad \text{and} \quad \Phi^*_\mu \to \Phi^{*\prime}_\mu = \Phi^*_\mu\vartheta^\dagger. \tag{2.9}$$

Therefore, a covariant derivative can be defined as

$$\begin{aligned} D_\mu\Phi &\equiv \Phi(\overleftarrow{\partial}_\mu + V^\dagger_\mu), \\ D_\mu\Phi^*_\nu &\equiv \Phi^*_\nu(\overleftarrow{\partial}_\mu + V^\dagger_\mu). \end{aligned} \tag{2.10}$$

Such a complication as the multiplication of the field $V^\dagger_\mu$ from the right hand side is due to the anti-doublet structure of the heavy meson fields Φ and Φ^*_μ.

Given the above definitions, one can write down the chirally invariant Lagrangian for Φ and Φ^*_μ with their couplings to the Goldstone bosons. Up to the first derivatives acting on the Goldstone boson fields, it has the form [19]

$$
\begin{aligned}
\mathcal{L} = \mathcal{L}_M^{SM} &+ D_\mu \Phi D^\mu \Phi^\dagger - m_\Phi^2 \Phi\Phi^\dagger - \tfrac{1}{2}\Phi^{*\mu\nu}\Phi^{*\dagger}_{\mu\nu} + m_{\Phi^*}^2 \Phi^{*\mu}\Phi^{*\dagger}_\mu \\
&+ f_Q(\Phi A^\mu \Phi^{*\dagger}_\mu + \Phi^*_\mu A^\mu \Phi^\dagger) + \tfrac{1}{2} g_Q \varepsilon^{\mu\nu\lambda\rho}(\Phi^*_{\mu\nu} A_\lambda \Phi^{*\dagger}_\rho + \Phi^*_\rho A_\lambda \Phi^{*\dagger}_{\mu\nu}),
\end{aligned}
\tag{2.11}
$$

where $\varepsilon_{0123} = +1$, f_Q and g_Q are the $\Phi^*\Phi M$ and $\Phi^*\Phi^* M$ coupling constants, respectively, and the field strength tensor is now newly defined in terms of the covariant derivative of Eq. (2.10) as

$$
\Phi^*_{\mu\nu} = D_\mu \Phi^*_\nu - D_\nu \Phi^*_\mu. \tag{2.12}
$$

In this work, instead of restricting the heavy meson masses, m_Φ and m_{Φ^*}, to their experimental values, we will take them as free parameters given by a formula

$$
\begin{aligned}
m_\Phi &= \overline{m}_\Phi - \frac{3\kappa}{4\overline{m}_\Phi}, \\
m_{\Phi^*} &= \overline{m}_\Phi + \frac{\kappa}{4\overline{m}_\Phi},
\end{aligned}
\tag{2.13}
$$

with the averaged meson mass $\overline{m}_\Phi \equiv \frac{1}{4}(m_\Phi + 3m_{\Phi^*})$. Given in Fig. 1 are the heavy meson masses evaluated by the formula (2.13) with the parameter κ fixed by the charmed meson masses m_D and m_{D^*}. One can see that the formula reproduces the heavy meson masses remarkably well not only in the bottom sector but also in the strangeness sector.

As for the coupling constants, since there is no experimental data available for them (except the upper limit $|g_Q|^2 < 0.5$ estimated via the D^* decay width and D^{*+} decay branching ratio), we will adopt the heavy-quark-symmetric relation [19]

$$
f_Q = 2 m_{\Phi^*} g_Q, \tag{2.14}
$$

and take the nonrelativistic quark model estimation for g_Q,

$$
g_Q = -0.75 \tag{2.15}
$$

over a wide range of heavy meson masses.

III. EQUATIONS OF MOTION FOR HEAVY MESON EIGENMODES

What interests us is the heavy meson eigenstates bound to the static potentials formed by the fields,

$$
\begin{aligned}
V^\mu &= (V^0, \mathbf{V}) = \big(0, -i(\boldsymbol{\tau}\times\hat{\mathbf{r}})\upsilon(r)\big), \\
A^\mu &= (A^0, \mathbf{A}) = \big(0, \tfrac{1}{2}[a_1(r)\boldsymbol{\tau} + a_2(r)\hat{\mathbf{r}}\boldsymbol{\tau}\cdot\hat{\mathbf{r}}]\big),
\end{aligned}
\tag{3.1}
$$

with $v(r) = [\sin^2(F/2)]/r$, $a_1(r) = (\sin F)/r$, and $a_2(r) = F' - (\sin F)/r$, (Hereafter, we will denote the derivative with respect to r and t by a prime and a dot, respectively. That is, $f' \equiv df/dr$ and $\dot{f} \equiv \partial_0 f$.) which are provided by the soliton configuration (2.2). We shall work in the soliton-fixed frame, neglecting any recoil effects of finite mass soliton. In Sec. V, we will roughly estimate such effects.

The equations of motion for the heavy mesons can be read off from the Lagrangian (2.11) as

$$D_\mu D^\mu \Phi^\dagger + m_\Phi^2 \Phi^\dagger = f_Q A^\mu \Phi_\mu^{*\dagger}, \tag{3.2}$$
$$D_\mu \Phi^{*\mu\nu\dagger} + m_{\Phi^*}^2 \Phi^{*\nu\dagger} = -f_Q A^\nu \Phi^\dagger + g_Q \varepsilon^{\mu\nu\lambda\rho} A_\lambda \Phi_{\mu\rho}^{*\dagger}. \tag{3.3}$$

In order to avoid any unnecessary complications originating from the anti-doublet structure of Φ and Φ_μ^*, we will work with $\Phi^\dagger$ and $\Phi_\mu^{*\dagger}$ instead.

An auxiliary condition corresponding to the Lorentz condition $\partial^\mu \Phi_\mu^* = 0$ for the free Proca equations can be obtained by taking a covariant divergence of Eq. (3.3):

$$m_{\Phi^*}^2 D_\nu \Phi^{*\nu\dagger} = -D_\nu D_\mu \Phi^{*\mu\nu\dagger} - f_Q D_\nu (A^\nu \Phi^\dagger) + g_Q \varepsilon^{\mu\nu\lambda\rho} D_\nu (A_\lambda \Phi_{\mu\rho}^{*\dagger}). \tag{3.4}$$

Note that none of the terms in the right hand side vanishes identically and that the auxiliary condition, thus, *cannot* be simply reduced to $D^\mu \Phi_\mu^* = 0$. Thus, different from the free Proca equations for the spin-1 field, it is very difficult to eliminate, for example, Φ_0^* by using Eq. (3.4) at this level. Actually, such an elimination is not an indispensable process in solving out the Proca equations as we shall see in the next section.

In terms of $\Phi^{*\mu} = (\Phi^{*0}, \boldsymbol{\Phi}^*)$ and using the form of (3.1), Eqs. (3.2-3.4) can be rewritten explicitly as

$$\ddot{\Phi}^\dagger - \mathbf{D}^2 \Phi^\dagger + m_\Phi^2 \Phi^\dagger = -f_Q \mathbf{A} \cdot \boldsymbol{\Phi}^{*\dagger}, \tag{3.5}$$
$$(\mathbf{D}^2 - m_{\Phi^*}^2) \Phi_0^{*\dagger} = \mathbf{D} \cdot \dot{\boldsymbol{\Phi}}^{*\dagger} - 2 g_Q \mathbf{A} \cdot (\mathbf{D} \times \boldsymbol{\Phi}^{*\dagger}), \tag{3.6}$$
$$\ddot{\boldsymbol{\Phi}}^{*\dagger} + \mathbf{D} \times (\mathbf{D} \times \boldsymbol{\Phi}^{*\dagger}) + m_{\Phi^*}^2 \boldsymbol{\Phi}^{*\dagger} = \mathbf{D} \dot{\Phi}_0^{*\dagger} - f_Q \mathbf{A} \Phi^\dagger - 2 g_Q \mathbf{A} \times (\dot{\boldsymbol{\Phi}}^{*\dagger} - \mathbf{D} \Phi_0^{*\dagger}), \tag{3.7}$$
$$\begin{aligned} & m_{\Phi^*}^2 (\dot{\Phi}_0^{*\dagger} - \mathbf{D} \cdot \boldsymbol{\Phi}^{*\dagger}) - \mathbf{D} \cdot [\mathbf{D} \times (\mathbf{D} \times \boldsymbol{\Phi}^{*\dagger})] \\ & \quad = f_Q \mathbf{D} \cdot (\mathbf{A} \Phi^\dagger) + 2 g_Q [\mathbf{D} \cdot \mathbf{A} \times (\dot{\boldsymbol{\Phi}}^{*\dagger} - \mathbf{D} \Phi_0^{*\dagger}) + \mathbf{A} \cdot \mathbf{D} \times \dot{\boldsymbol{\Phi}}^{*\dagger}], \end{aligned} \tag{3.8}$$

with $\mathbf{D} \equiv -\boldsymbol{\nabla} + \mathbf{V}$.

To solve them, we need to know the symmetries of the equations of motion. First of all, they are invariant under parity operations under which the heavy meson fields and the static soliton field transform as

$$\begin{array}{c} \Phi(\mathbf{r}, t) \longrightarrow -\Phi(-\mathbf{r}, t), \\ \Phi_0^*(\mathbf{r}, t) \longrightarrow +\Phi_0^*(-\mathbf{r}, t), \quad \boldsymbol{\Phi}^*(\mathbf{r}, t) \longrightarrow -\boldsymbol{\Phi}^*(-\mathbf{r}, t), \\ U(\mathbf{r}, t) \longrightarrow U^\dagger(-\mathbf{r}, t), \quad (\mathbf{V}, \mathbf{A} \longrightarrow -\mathbf{V}, +\mathbf{A}), \end{array} \tag{3.9}$$

where we have used that the heavy mesons have negative intrinsic parity. Next, due to the correlation of the isospin and angular momentum in the hedgehog configuration (2.2) and consequently in the static fields (3.1), the equations are only invariant under the "grand spin" rotation generated by the operator

$$\mathbf{K} = \mathbf{J} + \mathbf{I} = \mathbf{L} + \mathbf{S} + \mathbf{I}, \tag{3.10}$$

where $\mathbf{L}$, $\mathbf{S}$, and $\mathbf{I}$, respectively, denote the orbital angular momentum, spin, and isospin operator of the heavy mesons. (See Appendix A for their explicit forms and the corresponding eigenbases.) Thus, the eigenstates are classified by the grand spin quantum numbers (k, k_3) and the parity π.

For a given grand spin (k, k_3) with parity $\pi = (-1)^{k\pm 1/2}$, the general wavefunction of an energy eigenmode can be written as

$$\begin{aligned}
&\Phi^\dagger(\mathbf{r}, t) = e^{+i\omega t}\varphi(r)\mathcal{Y}^{(\pm)}_{kk_3}(\hat{\mathbf{r}}),\\
&\Phi^{*\dagger}_0(\mathbf{r}, t) = e^{+i\omega t} i\varphi^*_0(r)\mathcal{Y}^{(\mp)}_{kk_3}(\hat{\mathbf{r}}),\\
&\mathbf{\Phi}^{*\dagger}(\mathbf{r}, t) = e^{+i\omega t}\left[\varphi^*_1(r)\,\hat{\mathbf{r}}\mathcal{Y}^{(\mp)}_{kk_3}(\hat{\mathbf{r}}) + \varphi^*_2(r)\,\mathbf{L}\mathcal{Y}^{(\pm)}_{kk_3}(\hat{\mathbf{r}}) + \varphi^*_3(r)\,\mathbf{G}\mathcal{Y}^{(\mp)}_{kk_3}(\hat{\mathbf{r}})\right],
\end{aligned} \tag{3.11}$$

with five [20] radial functions $\varphi(r)$ and $\varphi^*_\alpha(r)$ $(\alpha = 0, 1, 2, 3)$. Here, $\mathbf{G}$ is an operator defined as $\mathbf{G} \equiv -i(\hat{\mathbf{r}} \times \mathbf{L})$, and $\mathcal{Y}^{(\pm)}_{kk_3}(\hat{\mathbf{r}})$ are the spinor spherical harmonics obtained by combining the orbital angular momentum eigenstates and the isospin eigenstates. (See Appendix A for further details.) Note the anomalous sign convention of the energy in the exponent for the time evolution of the eigenmodes. It is due to the fact that we are working with $\Phi^\dagger$ and $\Phi^{*\dagger}_\mu$ instead of Φ and Φ^*_μ; thus, the eigenenergy ω is positive for the bound states of heavy mesons and is negative for the "antiflavored" heavy mesons of $\bar{Q}q$ structure. Substituting Eq. (3.11) into Eqs. (3.5-3.7), we can obtain the equations of motion for the radial wavefunctions. Their explicit forms are given in Appendix B.

In the literatures, in order to reduce the complexity of the equations of motion, the solutions have been found approximately by adopting proper ans¨*atze*. In case of sufficiently heavy mesons, one may drop the $1/m_Q$ and higher order terms in Eq. (3.4) which leads us to a simple auxiliary condition [16]:

$$D^\mu \Phi^*_\mu = 0 \qquad \text{(Ansatz I)}. \tag{3.12}$$

It enables one to eliminate Φ^*_0 easily in favor of the other three fields $\mathbf{\Phi}^*$. In order to be consistent, one should also drop the higher order terms in the equations of motion for $\varphi^*_{1,2,3}$.

On the other hand, in the limit of light mesons one may use the fact that the vector mesons are *much heavier* than the pseudoscalar mesons. Then, the most dominant terms in the Lagrangian are those for the vector meson mass and the $\Phi\Phi^*M$ interactions with the coupling constant f_Q, which reads

$$\begin{aligned}
&m^2_{\Phi^*}\Phi^{*\mu}\Phi^{*\dagger}_\mu + f_Q(\Phi A^\mu \Phi^{*\dagger}_\mu + \Phi^*_\mu A^\mu \Phi^\dagger)\\
&\qquad = m^2_{\Phi^*}\left\{\Phi^{*\mu} + \varepsilon\Phi A^\mu\right\}\left\{\Phi^{*\dagger}_\mu + \varepsilon A_\mu \Phi^\dagger\right\} - 4g^2_Q \Phi A^\mu A_\mu \Phi^\dagger,
\end{aligned} \tag{3.13}$$

with ε abbreviating $2g_Q/m_{\Phi^*}$. Here, we have used the relation (2.14) for the coupling constant f_Q. Equation (3.13) suggests us another ansatz for the vector mesons [6]:

$$\Phi^{*\dagger}_\mu = -\frac{2g_Q}{m_{\Phi^*}} A_\mu \Phi^\dagger \qquad \text{(Ansatz II)}. \tag{3.14}$$

In terms of the radial functions, it reads

$$\varphi_0^*(r) = 0, \quad \varphi_1^*(r) = \tfrac{1}{2}\varepsilon(a_1 + a_2)\varphi(r), \quad \varphi_2^*(r) = \tfrac{1}{2}\varepsilon\gamma_{\mp}a_1\varphi(r) = -\varphi_3^*(r). \tag{3.15}$$

However, the equation of motion for the field Φ (or φ) should not be the one obtained simply by substituting Eq. (3.14) or (3.15) into the corresponding equation to eliminate Φ_μ^*. Under this ansatz, the Lagrangian (2.11) is reduced to that of the pseudoscalar field Φ only:

$$\begin{aligned}\mathcal{L}_\Phi = {} & D_\mu\Phi D^\mu\Phi^\dagger - m_\Phi^2\Phi\Phi^\dagger - \varepsilon^2 D_\mu\Phi A_\nu(A^\nu D^\mu\Phi^\dagger - A^\mu D^\nu\Phi^\dagger) \\ & -4g_Q^2\Phi A_\mu A^\mu\Phi^\dagger + g_Q\varepsilon^2\varepsilon^{\mu\nu\lambda\rho}(D_\mu\Phi A_\nu A_\lambda A_\rho\Phi^\dagger + \Phi A_\rho A_\lambda A_\nu D_\mu\Phi^\dagger),\end{aligned} \tag{3.16}$$

which yields the equation of motion for Φ as

$$\begin{aligned}(1+\varepsilon^2\mathbf{A}\cdot\mathbf{A})\ddot{\Phi}^\dagger + 2g_Q\varepsilon^2\mathbf{A}\cdot(\mathbf{A}\times\mathbf{A})\dot{\Phi}^\dagger - \mathbf{D}^2\Phi^\dagger + (m_\Phi^2 - 4g_Q^2\mathbf{A}\cdot\mathbf{A})\Phi^\dagger \\ -\varepsilon^2(\mathbf{A}\times\mathbf{D})\cdot(\mathbf{A}\times\mathbf{D})\Phi^\dagger = 0.\end{aligned} \tag{3.17}$$

Substituting $\Phi^\dagger(\mathbf{r},t) = e^{i\omega t}\varphi(r)\mathcal{Y}^\pm(\hat{\mathbf{r}})$, we can obtain the equation of motion for $\varphi(r)$ as

$$h(r)\varphi'' + \tilde{h}'\varphi' + [f(r)\omega^2 + \lambda(r)\omega - m_\Phi^2 - V_{\text{eff}}(r)]\varphi = 0, \tag{3.18}$$

where

$$\begin{aligned}& h(r) = 1 + \tfrac{1}{2}\varepsilon^2 a_1^2, \quad \tilde{h}'(r) = \frac{1}{r^2}[r^2 h(r)]', \\ & f(r) = 1 + \tfrac{1}{4}\varepsilon^2[(a_1+a_2)^2 + 2a_1^2], \\ & \lambda(r) = \tfrac{3}{2}\varepsilon^2 g_Q a_1^2(a_1+a_2), \\ & V_{\text{eff}}(r) = \frac{\lambda_\pm}{r^2} + \frac{2\mu_\pm}{r}v + 2v^2 - g_Q^2[(a_1+a_2)^2 + 2a_1^2] \\ & \qquad + \tfrac{1}{4}\varepsilon^2\left\{\left(\frac{\mu_\pm}{r} + 2v\right)^2 a_1^2 + a_1(a_1' + a_2')\left(\frac{\mu_\pm}{r} + 2v\right) + (a_1+a_2)^2\left(\tfrac{3}{2}a_1^2 + \frac{\lambda_\mp + \mu_\mp}{r^2}\right)\right\}.\end{aligned} \tag{3.19}$$

The similarity of the equation to those of Refs. [1,6] is remarkable. Those ε-dependent terms play an important role in obtaining the bound states in the light meson mass limit. Note that the factor ε is not necessarily small for the Ansatz II to be a good approximation [21]. In the strangeness sector, for example, we have $\varepsilon \sim -0.85$ [in $(ef_\pi)^{-1}$ unit].

However, as the mass of the involving heavy flavor increases, the mass of the pseudoscalar mesons becomes comparable to that of the vector mesons, and Ansatz II cannot be expected to work well. Note that it suppresses the role of the vector mesons by the factor ε, while the heavy quark symmetry implies that they are not distinguishable from the pseudoscalars as far as the low energy strong interactions are concerned. In Sec. V, we will give the exact solutions with the approximated ones using (3.12) and (3.14).

IV. SOLUTIONS TO FREE EQUATIONS

Before proceeding, we digress for a while to get an insight from the *spherical wave solutions* to the *free* equations, which are obtained by inserting (3.11) into $\mathcal{L}^{\rm free}$ of (2.4):

$$\varphi'' + \frac{2}{r}\varphi' - (m_\Phi^2 - \omega^2 + \frac{\lambda_\pm}{r^2})\varphi = 0, \tag{4.1}$$

$$\varphi_0^{*\prime\prime} + \frac{2}{r}\varphi_0^{*\prime} - (m_{\Phi^*}^2 + \frac{\lambda_\mp}{r^2})\varphi_0^* = -\omega[(\varphi_1^{*\prime} + \frac{2}{r}\varphi_1^*) - \frac{\lambda_\mp}{r}\varphi_3^*], \tag{4.2}$$

$$(m_{\Phi^*}^2 - \omega^2 + \frac{\lambda_\mp}{r^2})\varphi_1^* = \omega\varphi_0^{*\prime} + \frac{\lambda_\mp}{r}(\varphi_3^{*\prime} + \frac{1}{r}\varphi_3^*), \tag{4.3}$$

$$\varphi_2^{*\prime\prime} + \frac{2}{r}\varphi_2^{*\prime} - (m_{\Phi^*}^2 - \omega^2 + \frac{\lambda_\pm}{r^2})\varphi_2^* = 0, \tag{4.4}$$

$$\varphi_3^{*\prime\prime} + \frac{2}{r}\varphi_3^{*\prime} - (m_{\Phi^*}^2 - \omega^2)\varphi_3^* = \frac{1}{r}\varphi_1^{*\prime} - \frac{1}{r}\omega\varphi_0^*, \tag{4.5}$$

and the Lorentz condition

$$\omega\varphi_0^* = (\varphi_1^{*\prime} + \frac{2}{r}\varphi_1^*) - \frac{\lambda_\mp}{r}\varphi_3^*. \tag{4.6}$$

For a comparison, we are insisting to find the solutions in the form of Eq. (3.11). In case of free fields, $\varphi(r)$ and $\varphi_2^*(r)$ are completely decoupled to the other radial functions and Eqs. (4.1) and (4.4) lead to the solutions as

$$\varphi(r) = j_{\ell_\pm}(qr) \quad \text{and} \quad \varphi_2^*(r) = j_{\ell_\pm}(q^* r), \tag{4.7}$$

where $j_{\ell_\pm}(x)$ is the spherical Bessel function of order $\ell_\pm(= k \mp 1/2)$ with $q^2 \equiv \omega^2 - m_\Phi^2$ and $q^{*2} = \omega^2 - m_{\Phi^*}^2$. That is, we have found two of *four* independent solution sets of the equations:

$$\text{Solution I}: \begin{cases} \Phi^\dagger(\mathbf{r},t) = e^{i\omega t} j_{\ell_\pm}(qr)\mathcal{Y}^{(\pm)}_{k\,m_k}(\hat{\mathbf{r}}), \\ \Phi^{*\dagger}_\mu(\mathbf{r},t) = 0, \end{cases} \tag{4.8}$$

$$\text{Solution II}: \begin{cases} \Phi^\dagger(\mathbf{r},t) = 0, \\ \Phi^{*\dagger}_0(\mathbf{r},t) = 0, \\ \mathbf{\Phi}^{*\dagger}(\mathbf{r},t) = e^{i\omega t} j_{\ell_\pm}(q^* r)\,\mathbf{L}\mathcal{Y}^{(\pm)}_{k\,m_k}(\hat{\mathbf{r}}), \end{cases} \tag{4.9}$$

up to some proper normalizations. Note that there is no conjugate field of Φ_0^* in the "Solution II," which is consistent with the Lorentz condition (4.6).

The other two solution sets can be found by the following way. With the help of the Lorentz condition (4.6), Eqs. (4.2), (4.4), and (4.5) can be rewritten as

$$\varphi_0^{*\prime\prime} + \frac{2}{r}\varphi_0^{*\prime} + (q^{*2} - \frac{\lambda_\mp}{r^2})\varphi_0^* = 0, \tag{4.10}$$

$$\varphi_1^{*\prime\prime} + \frac{2}{r}\varphi_1^{*\prime} + (q^{*2} - \frac{\lambda_\mp + 2}{r^2})\varphi_1^* + \frac{2\lambda_\mp}{r^2}\varphi_3^* = 0, \tag{4.11}$$

$$\varphi_3^{*\prime\prime} + \frac{2}{r}\varphi_3^{*\prime} + (q^{*2} - \frac{\lambda_\mp}{r^2})\varphi_3^* + \frac{2}{r^2}\varphi_1^* = 0. \tag{4.12}$$

By diagonalizing the φ_1^*-φ_3^* mixing part as

$$\begin{pmatrix} \lambda_\mp + 2 & -2 \\ -2\lambda_\mp & \lambda_\mp \end{pmatrix} \begin{pmatrix} \varphi_1^* \\ \varphi_3^* \end{pmatrix} = \ell'_\pm(\ell'_\pm + 1) \begin{pmatrix} \varphi_1^* \\ \varphi_3^* \end{pmatrix}, \tag{4.13}$$

we can obtain the two remaining independent solutions as

$$\text{Solution III} : \begin{cases} \varphi_0^*(r) = (k+1/2)(q^*/\omega) j_{k+1/2}(q^* r), \\ \varphi_1^*(r) = (k+1/2) j_{k-1/2}(q^* r), \\ \varphi_3^*(r) = j_{k-1/2}(q^* r), \end{cases} \tag{4.14}$$

$$\text{Solution IV} : \begin{cases} \varphi_0^*(r) = (k+3/2)(q^*/\omega) j_{k+1/2}(q^* r), \\ \varphi_1^*(r) = (k+3/2) j_{k+3/2}(q^* r), \\ \varphi_3^*(r) = -j_{k+3/2}(q^* r), \end{cases} \tag{4.15}$$

for the $\pi = (-1)^{k+1/2}$ states and

$$\text{Solution III} : \begin{cases} \varphi_0^*(r) = (k+1/2)(q^*/\omega) j_{k-1/2}(q^* r), \\ \varphi_1^*(r) = (k+1/2) j_{k+1/2}(q^* r), \\ \varphi_3^*(r) = -j_{k+1/2}(q^* r), \end{cases} \tag{4.16}$$

$$\text{Solution IV} : \begin{cases} \varphi_0^*(r) = (k-1/2)(q^*/\omega) j_{k-1/2}(q^* r), \\ \varphi_1^*(r) = (k-1/2) j_{k-3/2}(q^* r), \\ \varphi_3^*(r) = j_{k-3/2}(q^* r), \end{cases} \tag{4.17}$$

for the $\pi = (-1)^{k-1/2}$ states.

From these free solutions we can learn the followings:

1. In principle, we can obtain the solutions in the form of Eq. (3.11) without eliminating one of Φ_μ^* at the level of Eqs. (3.5-3.8). It is enough to keep in mind that, due to the dependence in field variables (and consequently in the equations of motion), we can obtain only four independent solutions.

2. The φ_1^*-φ_3^* mixing is due to the fact that our grand spin eigenstates, $\hat{\mathbf{r}}\mathcal{Y}_{k\,m_k}^{(\pm)}$ and $\mathbf{G}\mathcal{Y}_{k\,m_k}^{(\pm)}$, are not the eigenstates of the orbital angular momentum $\mathbf{L}^2$. If we have used $\boldsymbol{\mathcal{Y}}_{k\,m_k}^{(i)}$, the conjugate radial functions would have been decoupled. For example, the solution III of Eq. (4.14) can be written in a form of

$$\text{Solution III} : \begin{cases} \Phi^\dagger(\mathbf{r}, t) = 0, \\ \Phi_0^{*\dagger}(\mathbf{r}, t) = i(k+1/2)(q^*/\omega) j_{k+1/2}(q^* r) \mathcal{Y}_{k\,m_k}^{(-)}(\hat{\mathbf{r}}), \\ \boldsymbol{\Phi}^{*\dagger}(\mathbf{r}, t) = j_{k-1/2}(q^* r) \left[(k+1/2)\, \hat{\mathbf{r}}\mathcal{Y}_{k\,m_k}^{(-)}(\hat{\mathbf{r}}) + \mathbf{G}\mathcal{Y}_{k\,m_k}^{(-)}(\hat{\mathbf{r}}) \right] \\ \qquad\qquad = \sqrt{(2k+1)(k+1)} j_{k-1/2}(q^* r)\, \boldsymbol{\mathcal{Y}}_{k\,m_k}^{(2)}(\hat{\mathbf{r}}). \end{cases} \tag{4.18}$$

3. The special form of Eq. (4.3) and (B3) with no derivative on φ_1^* is also ascribed to our choice of the grand spin eigenstates. Although one can easily express φ_1^* in terms of the other fields by using the equation

$$\varphi_1^* = -\frac{\omega\varphi_0^{*\prime} + \lambda_\mp/r(\varphi_3^{*\prime} + \varphi_3^*/r)}{\omega^2 - m_{\Phi^*}^2 - \lambda_\mp/r^2}, \tag{4.19}$$

it cannot be used in eliminating φ_1^* from the equations. Let's see what happens if one proceeds along this direction. Substituting Eq. (4.19) and its derivative into Eqs. (4.2) and (4.5) leads to

$$\begin{aligned} \frac{m_{\Phi^*}^2 + \lambda_\mp/r^2}{\omega^2 - m_{\Phi^*}^2 - \lambda_\mp/r^2}\,\varphi_0^{*\prime\prime} + \frac{\omega\lambda_\mp/r}{\omega^2 - m_{\Phi^*}^2 - \lambda_\mp/r^2}\,\varphi_3^{*\prime\prime} &= \tilde{p}_0, \\ \frac{\omega/r}{\omega^2 - m_{\Phi^*}^2 - \lambda_\mp/r^2}\,\varphi_0^{*\prime\prime} + \frac{\omega^2 - m_{\Phi^*}^2}{\omega^2 - m_{\Phi^*}^2 - \lambda_\mp/r^2}\,\varphi_3^{*\prime\prime} &= \tilde{p}_3, \end{aligned} \tag{4.20}$$

where $\tilde{p}_{0,3}$ are the terms without containing the second order derivatives. In order to obtain $\varphi_{0,3}^{*\prime\prime}$, we should solve the above linear equations. However, the matrix of the coefficients is singular at the origin and, thus, one cannot obtain the correct solutions in this way.

V. NUMERICAL SOLUTIONS AND DISCUSSIONS

In solving equations numerically, there is a considerable freedom. We will solve Eqs. (B1) and (B3-B6) by taking φ, φ', φ_0^*, φ_1^*, φ_2^*, $\varphi_2^{*\prime}$, φ_3^*, and $\varphi_3^{*\prime}$ as *eight* independent variables. Note that Eqs. (B1), (B4), and (B5) are the second order differential equations for φ, φ_2^*, and φ_3^*, while Eqs. (B3) and (B6) are the first order ones for φ_0^* and φ_1^*, respectively. On the other hand, we can take Eqs. (B1), (B2), (B4), and (B5) as the second order differential equations for φ, φ_0^*, φ_2^*, and φ_3^* and use Eqs. (B3) and (B6) to eliminate φ_1^* and $\varphi_1^{*\prime}$ appearing in those equations. One may be tempted to eliminate φ_1^* by using only Eq. (B3), since it does not contain any derivative on φ_1^*. However, the numerical program according to such a recipe meets serious instabilities near the origin due to the singularity in a linear equation similar to Eq. (4.20).

In Fig. 2, we give the radial wavefunctions for a few eigenstates (solid lines). For a comparison, we present the approximate solutions obtained by using Ansatz I (dashed lines) and Ansatz II (dash-dotted lines) as well. In case of the Ansatz II, $\varphi_{1,2}^*$ are obtained from the relations (3.15) with the solution of Eq. (3.18) substituted for φ. As we have expected, Ansatz II works only in the limit case; the approximate radial functions of Ansatz II are quite close to the exact ones in the strangeness sector but it becomes a bad approximation in the charm sector. One can see that the factor ε of the Ansatz II suppresses the vector fields too much. On the other hand, Ansatz I works quite well both in the strangeness sector and in the charm sector. In the bottom sector, the approximate solutions are indistinguishable from the exact ones. This observation supports the approximation of Ref. [16].

From Fig. 3, one may arrive at the same conclusion for the Ans$\ddot{}$*atze* I and II, where we present the eigenenergies of the $k^\pi{=}\frac{1}{2}^+$ ground state, $\frac{1}{2}^+$ radially excited state, and $\frac{1}{2}^-$ state as functions of the heavy pseudoscalar meson mass m_Φ. That is, Ansatz I works well over a rather wide range of the heavy meson masses while Ansatz II works well only in the limit cases, say, $m_\Phi < 0.8$ GeV. Therefore, the Ansatz II can be justified only in the strangeness

sector as in Ref. [6]. Note that the Ansatz I and II could yield *lower* eigenenergies than the exact ones. However, it does not contradict our common sense that the exact solutions should have the lower eigenenergy than the approximate ones. Through the ans¨atze we have altered the interactions of the heavy mesons with the soliton more or less. Note also that the equations admit one bound state in the strangeness sector without the higher derivative terms. In case of Ansatz II, the ε terms are essential for the existence of the bound state.

Given in Fig. 4 are the *binding energies* $\omega - m_\Phi$ of the bound states as functions of the heavy pseudoscalar meson mass m_Φ. As the heavy meson mass increases, the $k^\pi = \frac{1}{2}^+$ ground state energy decreases and it may reach down to the infinite mass limit $-\frac{3}{2}g_Q F'(0) \sim 0.8$ GeV. However, the kinetic effect turns out to be quite large and the binding energy does not come to the infinite mass limit value even when the heavy meson mass is increased up to 10 GeV. Note that, as we have expected from the heavy quark spin symmetry, the eigenstates of $k = k_\ell \pm 1/2$ become degenerate as the heavy meson mass increases. A special care should be taken in the quantization of these degenerate bound states. [14] In Fig. 4, there appear many bound states for the heavy mesons in the charm and bottom sector. However, remind that we have neglected any recoil effects due to the finite soliton mass. At this point, we may *naïvely* estimate the effects of the soliton motion by replacing the heavy meson masses with their corresponding "reduced" masses defined by

$$\mu_\Phi \equiv \frac{m_\Phi m_{\rm sol}}{m_\Phi + m_{\rm sol}}, \tag{5.1}$$

where $m_{\rm sol}$ is the soliton mass. With $m_{\rm sol} = 867$ MeV, we have $\mu_D \sim 590$ MeV and $\mu_B \sim 743$ MeV. From Fig. 4, one can see that only one (or at most two) bound state(s) could survive when the recoil effects are incorporated. This correction deserves to be studied further.

As discussed in Refs. [14,15], the equations admit the bound state solutions with negative eigenenergies, which can be interpreted as the bound states of antiflavored heavy mesons. In Fig. 5, we present the radial functions of a few low-lying bound states in case of $\bar{Q} = \bar{b}$. It is apparent that the radial functions spread over a wider range of r than those of $Q = b$ do. The binding energies of antiflavored heavy mesons are given in Fig. 6 as functions of the heavy meson mass. In this case, the energy change due to the recoil effects is comparable to the binding energy and the recoil effects seem to be crucial for the existence of the bound state. However, a naïve estimation using (5.1) shows that there still survive a bound state near the threshold, which leaves a possibility of stable pentaquark baryons.

As a summary, we obtained the energy levels of the soliton–heavy-meson bound states. The equations of motion are solved exactly in a given model Lagrangian for the excited states as well as for the ground state. The calculation was also made for the pentaquark exotic baryons. However, to obtain the real mass spectrum of heavy baryons, we should go one step further; i.e., the system should be quantized for describing baryons of definite spin and isospin quantum numbers. Work in this direction is in progress and will be reported elsewhere.

Note added. After completion of this paper, we were aware of recent work of Schechter and Subbaraman [25]. In estimating the kinetic effects of heavy mesons, the authors adopt an interesting approximation to the equations of motion. Corrections due to finite soliton mass are roughly calculated in the same way as in this paper.

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APPENDIX A: GRAND SPIN EIGENSTATES

In this Appendix, we construct the grand spin eigenstates, i.e., the angular part of the wavefunction (3.11), by combining the orbital angular momentum (**L**), spin (**S**), and isospin (**I**) of the heavy mesons. We first combine the orbital angular momentum and spin to get the angular momentum (**J**) eigenstates, and then combine the isospin [22].

To do this, we first find the spin operator S_i $(i = 1, 2, 3)$ for the vector mesons and the corresponding eigenstates. The Lagrangian (2.4) is invariant under an infinitesimal Lorentz transformation

$$\begin{aligned}
&x^\mu \to x'^\mu = x^\mu + \epsilon^\mu{}_\nu x^\nu, \\
&\Phi(x) \to \Phi'(x') = \Phi(x), \\
&\Phi^*_\alpha(x) \to \Phi'^*_\alpha(x') = \tfrac{1}{2}\epsilon^{\mu\nu}(S_{\mu\nu})_\alpha{}^\beta \Phi^*_\beta(x),
\end{aligned} \tag{A1}$$

where $\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}$ and $(S^{\mu\nu})_{\alpha\beta} = g^\mu{}_\alpha g^\nu{}_\beta - g^\mu{}_\beta g^\nu{}_\alpha$ with $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. It defines conserved angular momentum

$$\begin{aligned}
&J^i = \tfrac{1}{2}\varepsilon^{ijk}\int d^3r \mathcal{M}_{0jk}, \\
&\mathcal{M}_{0jk} = (x^j\mathcal{P}^{0k} - x^k\mathcal{P}^{0j}) + [\Pi^{*m}(S^{kj})_{mn}\Phi^{*n\dagger} + \Phi^{*m}(S^{kj})_{mn}\Pi^{*n\dagger}],
\end{aligned} \tag{A2}$$

where $\mathcal{P}^{\mu\nu}$ is the canonical energy-momentum tensor and $\Pi^{*n}(\equiv \partial\mathcal{L}^{\rm free}_\Phi/\partial\dot{\Phi}^*_n)$ is the momentum conjugate to the field Φ^*_n. Here, the indices run from 1 to 3. The first part corresponds to the orbital angular momentum **L** and the second to the spin angular momentum **S** of the vector mesons. The latter can be rewritten neatly as

$$\mathbf{S} = \int d^3r\ (\mathbf{\Pi}^* \times \mathbf{\Phi}^{*\dagger} + \mathbf{\Phi}^* \times \mathbf{\Pi}^{*\dagger}), \tag{A3}$$

which defines the corresponding quantum mechanical spin operator acting on the vector meson wavefunctions as

$$S_1 = i\hat{\mathbf{e}}_1\times, \quad S_2 = i\hat{\mathbf{e}}_2\times, \quad S_3 = i\hat{\mathbf{e}}_3\times, \tag{A4}$$

where $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, and $\hat{\mathbf{e}}_3$ are unit vectors along the x, y, and z axis, respectively. Strictly speaking, although they satisfy the correct commutation relations $[S_i, S_j] = i\varepsilon_{ijk}S_k$, and the square is an invariant of the group, it is not the relativistic spin operator that commutes with all the generators of the Lorentz group. It is a good spin operator only in the rest frame, which is, however, enough for us to proceed. Actually, the invariance of the equations (3.5-3.8) under the grand spin rotation can be achieved only with this spin operator.

Eigenvectors of $\mathbf{S}^2$ and S_3 are found by taking suitable linear combinations of the unit vectors $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, and $\hat{\mathbf{e}}_3$ [23]. Let's define

$$\begin{aligned} \hat{\mathbf{e}}_{\pm 1} &= \mp \tfrac{1}{\sqrt{2}}(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2), \\ \hat{\mathbf{e}}_0 &= \hat{\mathbf{e}}_3, \end{aligned} \tag{A5}$$

which satisfy

$$\begin{aligned} \mathbf{S}^2\,\hat{\mathbf{e}}_q &= 2\,\hat{\mathbf{e}}_q, \\ S_3\,\hat{\mathbf{e}}_q &= q\,\hat{\mathbf{e}}_q, \end{aligned} \qquad \text{with } q = \pm 1, 0. \tag{A6}$$

Then, the eigenvectors of $\mathbf{J}^2$ and J_3 for the pseudoscalar mesons are simply the spherical harmonics $Y_{jm_j}(\hat{\mathbf{r}})$, and for the vector mesons the "vector spherical harmonics"

$$\mathbf{Y}_{j\,\ell\,m_j}(\hat{\mathbf{r}}) = \sum_{m,q} Y_{\ell\,m}(\hat{\mathbf{r}})\,\hat{\mathbf{e}}_q\,(\ell\,m\,1\,q|j\,m_j), \tag{A7}$$

where $(\ell m 1 q|j m_j)$ is the Clebsch-Gordan coefficient of adding the orbital angular momentum and the spin. As a result of the angular momentum addition rule, there can be three different kinds of vector spherical harmonics with a given (j, m_j):

$$\begin{aligned} &\mathbf{Y}_{j\,j\,m_j}(\hat{\mathbf{r}}) \quad \text{with parity } \pi = -(-1)^j\,, \\ &\mathbf{Y}_{j\,j\pm 1\,m_j}(\hat{\mathbf{r}}) \quad \text{with parity } \pi = +(-1)^j, \end{aligned} \tag{A8}$$

where the parity π incorporates the intrinsic parity of the vector mesons.

The vector spherical harmonics can be generated from the spherical harmonics $Y_{\ell m}(\hat{\mathbf{r}})$ by making use of certain operators: [24]

$$\begin{aligned} \mathbf{Y}_{j\,j\,m_j} &= \frac{1}{\sqrt{j(j+1)}}\,\mathbf{L}Y_{j\,m_j}, \\ \mathbf{Y}_{j\,j+1\,m_j} &= \sqrt{\frac{j}{2j+1}}\frac{1}{\sqrt{j(j+1)}}\,\mathbf{G}Y_{j\,m_j} - \sqrt{\frac{j+1}{2j+1}}\,\hat{\mathbf{r}}Y_{j\,m_j}, \\ \mathbf{Y}_{j\,j-1\,m_j} &= \sqrt{\frac{j+1}{2j+1}}\frac{1}{\sqrt{j(j+1)}}\,\mathbf{G}Y_{j\,m_j} + \sqrt{\frac{j}{2j+1}}\,\hat{\mathbf{r}}Y_{j\,m_j}, \end{aligned} \tag{A9}$$

where $\mathbf{G} = -i(\hat{\mathbf{r}} \times \mathbf{L})$. It enables us to carry out somewhat tedious calculations involving vector spherical harmonics through elementary vector algebra without referring Clebsch-Gordan coefficients.

To complete the job, we combine the isospin to these angular momentum eigenstates. The quantum mechanical isospin operator acting on the *isodoublet* (not anti-doublet) structure $\Phi^\dagger$ and $\Phi_\mu^{*\dagger}$ is given by the Pauli matrices as

$$\mathbf{I} = \tfrac{1}{2}\boldsymbol{\tau}. \tag{A10}$$

We denote the eigenstates of $\mathbf{I}^2$ and I_3 as

$$\chi_{+\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \chi_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{A11}$$

which have the properties $\mathbf{I}^2 \chi_{\pm\frac{1}{2}} = \frac{3}{4}\chi_{\pm\frac{1}{2}}$ and $I_3 \chi_{\pm\frac{1}{2}} = \pm\frac{1}{2}\chi_{\pm\frac{1}{2}}$.

Now, for the pseudoscalar mesons, the grand spin eigenstates are obtained simply by the spinor spherical harmonics:

$$\begin{aligned}
&\text{(i) for } k = \ell + 1/2 \text{ and } \pi = (-1)^{k+1/2}, \\
&\quad \mathcal{Y}^{(+)}_{k\,m_k}(\hat{\mathbf{r}}) = +\sqrt{\frac{k+m_k}{2k}}\, Y_{\ell\, m_k - 1/2}\, \chi_{+\frac{1}{2}} + \sqrt{\frac{k-m_k}{2k}}\, Y_{\ell\, m_k + 1/2}\, \chi_{-\frac{1}{2}}, \\
&\text{(ii) for } k = \ell - 1/2 \text{ and } \pi = (-1)^{k-1/2}, \\
&\quad \mathcal{Y}^{(-)}_{k\,m_k}(\hat{\mathbf{r}}) = -\sqrt{\frac{k-m_k+1}{2(k+1)}}\, Y_{\ell\, m_k - 1/2}\, \chi_{+\frac{1}{2}} + \sqrt{\frac{k+m_k+1}{2(k+1)}}\, Y_{\ell\, m_k + 1/2}\, \chi_{-\frac{1}{2}},
\end{aligned} \tag{A12}$$

which are related with each other by

$$-\left(\boldsymbol{\tau} \cdot \hat{\mathbf{r}}\right) \mathcal{Y}^{(\pm)}_{k\,m_k} = \mathcal{Y}^{(\mp)}_{k\,m_k}. \tag{A13}$$

As for the grand spin eigenstates of the vector mesons, coupling of the isospin eigenstates to the vector spherical harmonics (A7) leads to six different grand spin eigenstates $\boldsymbol{\mathcal{Y}}^{(i)}_{k\,k_3}$ for a given (k, k_3):

$$\boldsymbol{\mathcal{Y}}^{(i)}_{k\,m_k}(\hat{\mathbf{r}}) = \sum_{m_j q} \mathbf{Y}_{j_i \ell_i m_j}(\hat{\mathbf{r}})\, \chi_q\, (j_i\, m_j\, \tfrac{1}{2}\, q | k\, m_k), \tag{A14}$$

with $k = j_i \pm 1/2$ and $j_i = \ell_i, \ell_i \pm 1$. We list possible j_i and ℓ_i in Table I. By the help of Eq. (A9), we can rewrite them in terms of $\hat{\mathbf{r}}\mathcal{Y}^{(\pm)}_{k\,m_k}$, $\mathbf{L}\mathcal{Y}^{(\pm)}_{k\,m_k}$, and $\mathbf{G}\mathcal{Y}^{(\pm)}_{k\,m_k}$, for example,

$$\begin{aligned}
\boldsymbol{\mathcal{Y}}^{(1)}_{k\,m_k} &= \frac{1}{\sqrt{j(j+1)}}\, \mathbf{L}\mathcal{Y}^{(+)}_{k\,m_k}, \\
\boldsymbol{\mathcal{Y}}^{(5)}_{k\,m_k} &= \sqrt{\frac{j+1}{2j+1}} \frac{1}{\sqrt{j(j+1)}}\, \mathbf{G}\mathcal{Y}^{(+)}_{k\,m_k} + \sqrt{\frac{j}{2j+1}}\, \hat{\mathbf{r}}\mathcal{Y}^{(+)}_{k\,m_k}, \\
\boldsymbol{\mathcal{Y}}^{(6)}_{k\,m_k} &= \sqrt{\frac{j}{2j+1}} \frac{1}{\sqrt{j(j+1)}}\, \mathbf{G}\mathcal{Y}^{(+)}_{k\,m_k} - \sqrt{\frac{j+1}{2j+1}}\, \hat{\mathbf{r}}\mathcal{Y}^{(+)}_{k\,m_k},
\end{aligned} \tag{A15}$$

with $j = k + 1/2$. For a practical convenience, we take $\hat{\mathbf{r}}\mathcal{Y}^{(\pm)}_{k\,m_k}$, $\mathbf{L}\mathcal{Y}^{(\pm)}_{k\,m_k}$, and $\mathbf{G}\mathcal{Y}^{(\pm)}_{k\,m_k}$ as six independent grand spin eigenstates instead of $\boldsymbol{\mathcal{Y}}^{(i)}_{k\,m_k}$.

The general solution to Eqs. (3.5-3.8) for a given (k, m_k) can now be written as

$$\begin{aligned}
\Phi^\dagger(\mathbf{r}, t) &= e^{+i\omega t}\left[\varphi^{(+)}(r)\mathcal{Y}^{(+)}_{k\,m_k}(\hat{\mathbf{r}}) + \varphi^{(-)}(r)\mathcal{Y}^{(-)}_{k\,m_k}(\hat{\mathbf{r}})\right], \\
\Phi_0^{*\dagger}(\mathbf{r}, t) &= i e^{+i\omega t}\left[\varphi_0^{*(-)}(r)\mathcal{Y}^{(+)}_{k\,m_k}(\hat{\mathbf{r}}) + \varphi_0^{*(+)}(r)\mathcal{Y}^{(-)}_{k\,m_k}(\hat{\mathbf{r}})\right], \\
\boldsymbol{\Phi}^{*\dagger}(\mathbf{r}, t) &= e^{+i\omega t}\Big[\varphi_1^{*(+)}(r)\, \hat{\mathbf{r}}\mathcal{Y}^{(-)}_{k\,m_k}(\hat{\mathbf{r}}) + \varphi_2^{*(+)}(r)\, \mathbf{L}\mathcal{Y}^{(+)}_{k\,m_k}(\hat{\mathbf{r}}) + \varphi_3^{*(+)}(r)\, \mathbf{G}\mathcal{Y}^{(-)}_{k\,m_k}(\hat{\mathbf{r}}) \\
&\qquad + \varphi_1^{*(-)}(r)\, \hat{\mathbf{r}}\mathcal{Y}^{(+)}_{k\,m_k}(\hat{\mathbf{r}}) + \varphi_2^{*(-)}(r)\, \mathbf{L}\mathcal{Y}^{(-)}_{k\,m_k}(\hat{\mathbf{r}}) + \varphi_3^{*(-)}(r)\, \mathbf{G}\mathcal{Y}^{(+)}_{k\,m_k}(\hat{\mathbf{r}})\Big].
\end{aligned} \tag{A16}$$

We may decompose them into two solution sets of definite parity as in Eq. (3.11).

APPENDIX B: EQUATIONS OF MOTION FOR RADIAL FUNCTIONS

In this Appendix, we give the explicit forms of the equations of motion for the radial functions $\varphi(r)$ and $\varphi^*_\alpha(r)$ $(\alpha = 0, 1, 2, 3)$. Substitution of Eq. (3.11) into the equations of motion (3.5-3.8) leads us to the following coupled differential equations,

$$\begin{aligned}
&\varphi'' + \frac{2}{r}\varphi' + \left[\omega^2 - m_\Phi^2 - \left(\frac{\lambda_\pm}{r^2} + \frac{2\upsilon}{r}\mu_\pm + 2\upsilon^2\right)\right]\varphi \\
&\quad = -\tfrac{1}{2} f_Q (a_1 + a_2)\varphi_1^* + \tfrac{1}{2} f_Q a_1 (\mu_\pm \varphi_2^* + \mu_\mp \varphi_3^*), && \text{(B1)} \\
&\varphi_0^{*\prime\prime} + \frac{2}{r}\varphi_0^{*\prime} - \left[m_{\Phi^*}^2 + \left(\frac{\lambda_\mp}{r^2} + \frac{2\upsilon}{r}\mu_\mp + 2\upsilon^2\right)\right]\varphi_0^* \\
&\quad = \omega\left[-\left(\varphi_1^{*\prime} + \frac{2}{r}\varphi_1^*\right) + \mu_\pm \upsilon \varphi_2^* + \left(\frac{\lambda_\mp}{r} + \mu_\mp \upsilon\right)\varphi_3^*\right] && \text{(B2)} \\
&\qquad + g_Q\left[(a_1 + a_2) f_1 - a_1(\mu_\mp f_2 + \mu_\pm f_3)\right], \\
&\left[m_{\Phi^*}^2 - \omega^2 + \left(\frac{\lambda_\mp}{r^2} + \frac{2\upsilon}{r}\mu_\mp + 2\upsilon^2\right)\right]\varphi_1^* \\
&\quad = \mu_\pm \upsilon\left(\varphi_2^{*\prime} + \frac{1}{r}\varphi_2^*\right) + \left(\frac{\lambda_\mp}{r} + \mu_\mp \upsilon\right)\left(\varphi_3^{*\prime} + \frac{1}{r}\varphi_3^*\right) + \omega\varphi_0^{*\prime} && \text{(B3)} \\
&\qquad + \tfrac{1}{2} f_Q (a_1 + a_2)\varphi - g_Q a_1 \left(\mu_\pm g_2 + \mu_\mp g_3\right), \\
&\varphi_2^{*\prime\prime} + \frac{2}{r}\varphi_2^{*\prime} + \left[\omega^2 - m_{\Phi^*}^2 - \left(\frac{\lambda_\pm}{r^2} + \frac{2\upsilon}{r}\mu_\pm + \gamma_\pm \mu_\pm \upsilon^2\right)\right]\varphi_2^* \\
&\quad = \gamma_\pm \upsilon \varphi_1^{*\prime} + \gamma_\pm\left(\upsilon' + \frac{\upsilon}{r}\right)\varphi_1^* + \mu_\mp \upsilon\left(\frac{1}{r} + \gamma_\pm \upsilon\right)\varphi_3^* - \omega\gamma_\pm \upsilon \varphi_0^* && \text{(B4)} \\
&\qquad + \tfrac{1}{2} f_Q a_1 \gamma_\pm \varphi + g_Q\left\{\gamma_\pm a_1 g_1 + (a_1 + a_2)\left[\gamma_\pm g_2 + (1 + \gamma_\pm) g_3\right]\right\}, \\
&\varphi_3^{*\prime\prime} + \frac{2}{r}\varphi_3^{*\prime} + \left[\omega^2 - m_{\Phi^*}^2 + \gamma_\pm \mu_\mp \upsilon^2\right]\varphi_3^* \\
&\quad = \left(\frac{1}{r} - \gamma_\pm \upsilon\right)\varphi_1^{*\prime} - \gamma_\pm\left(\upsilon' + \frac{\upsilon}{r}\right)\varphi_1^* - \mu_\pm \upsilon\left(\frac{1}{r} + \gamma_\pm \upsilon\right)\varphi_2^* - \omega\left(\frac{1}{r} - \gamma_\pm \upsilon\right)\varphi_0^* && \text{(B5)} \\
&\qquad - \tfrac{1}{2} f_Q a_1 \gamma_\pm \varphi + g_Q\left\{-\gamma_\pm a_1 g_1 + (a_1 + a_2)\left[(1 - \gamma_\pm) g_2 - \gamma_\pm g_3\right]\right\}, \\
&m_{\Phi^*}^2\left\{\omega\varphi_0^* - (\varphi_1^{*\prime} + \frac{2}{r}\varphi_1^*) + \mu_\pm \upsilon \varphi_2^* + (\frac{\lambda_\mp}{r} + \mu_\mp \upsilon)\varphi_3^*\right\} \\
&\quad + 2\upsilon\left(\frac{1}{r} - \upsilon\right) f_1 - \left(\upsilon' + \frac{\upsilon}{r}\right)\left[\mu_\mp f_2 + \mu_\pm f_3\right] \\
&= -\frac{f_Q}{2}\left[(a_1 + a_2)(\varphi' + \frac{2}{r}\varphi) + (a_1 + a_2)'\varphi + (\frac{\mu_\mp}{r} + 2\upsilon) a_1 \varphi\right] && \text{(B6)} \\
&\quad + 2 g_Q\left[a_1\left(\upsilon' + \frac{\upsilon}{r}\right) + \upsilon(a_1 + a_2)\left(\frac{1}{r} - \upsilon\right)\right]\varphi_0^*,
\end{aligned}$$

where f_i and g_i (i=1,2,3) are functions defined by

$$\mathbf{D}\times\mathbf{\Phi}^{*\dagger} \equiv i\left[f_1\,\hat{\mathbf{r}}\mathcal{Y}^{(\pm)}_{k\,m_k} + f_2\,\mathbf{L}\mathcal{Y}^{(\mp)}_{k\,m_k} + f_3\,\mathbf{G}\mathcal{Y}^{(\pm)}_{k\,m_k}\right], \tag{B7}$$

$$\dot{\mathbf{\Phi}}^{*\dagger} - \mathbf{D}\Phi_0^{*\dagger} \equiv i\left[g_1\,\hat{\mathbf{r}}\mathcal{Y}^{(\mp)}_{k\,m_k} + g_2\,\mathbf{L}\mathcal{Y}^{(\pm)}_{k\,m_k} + g_3\,\mathbf{G}\mathcal{Y}^{(\mp)}_{k\,m_k}\right], \tag{B8}$$

and $\lambda_\pm$, $\mu_\pm$, and $\gamma_\pm$ are the numbers defined through

$$\begin{aligned}
&\mathbf{L}^2\mathcal{Y}^{(\pm)}_{k\,m_k} \equiv \lambda_\pm\mathcal{Y}^{(\pm)}_{k\,m_k}, \qquad (\boldsymbol{\tau}\cdot\mathbf{L})\mathcal{Y}^{(\pm)}_{k\,m_k} \equiv \mu_\pm\mathcal{Y}^{(\pm)}_{k\,m_k}, \\
&\boldsymbol{\tau}\mathcal{Y}^{(\pm)}_{k\,m_k} \equiv -\hat{\mathbf{r}}\mathcal{Y}^{(\mp)}_{k\,m_k} + \gamma_\pm\,\mathbf{L}\mathcal{Y}^{(\pm)}_{k\,m_k} + \gamma_\mp\,\mathbf{G}\mathcal{Y}^{(\mp)}_{k\,m_k},
\end{aligned} \tag{B9}$$

which also give $\boldsymbol{\tau}\cdot\mathbf{G}\mathcal{Y}^{(\pm)}_{k\,m_k} = \mu_\pm\mathcal{Y}^{(\mp)}_{k\,m_k}$. Explicitly, f_i and g_i (i=1,2,3) are given in terms of φ^*_α (α=0,1,2,3) as

$$\begin{aligned}
f_1 &= -\left[\left(\frac{\lambda_\pm}{r} + \mu_\pm\upsilon\right)\varphi_2^* + \mu_\mp\upsilon\varphi_3^*\right], \\
f_2 &= +\left[\left(\frac{1}{r} - \gamma_\pm\upsilon\right)\varphi_1^* - \left(\varphi_3^{*\prime} + \frac{1}{r}\varphi_3^*\right)\right], \\
f_3 &= +\left[\gamma_\pm\upsilon\varphi_1^* - \left(\varphi_2^{*\prime} + \frac{1}{r}\varphi_2^*\right)\right],
\end{aligned} \tag{B10}$$

$$\begin{aligned}
g_1 &= \omega\varphi_1^* + \varphi_0^{*\prime}, \\
g_2 &= \omega\varphi_2^* + \gamma_\pm\upsilon\varphi_0^*, \\
g_3 &= \omega\varphi_3^* + \left(\frac{1}{r} - \gamma_\pm\upsilon\right)\varphi_0^*,
\end{aligned} \tag{B11}$$

and $\lambda_\pm$, $\mu_\pm$, and $\gamma_\pm$ are written in terms of k as

$$\begin{aligned}
&\lambda_+ = (k-1/2)(k+1/2), && \lambda_- = (k+1/2)(k+3/2), \\
&\mu_+ = k-1/2, && \mu_- = -(k+3/2), \\
&\gamma_+ = \mu_+/\lambda_+ = 1/(k+1/2), && \gamma_- = \mu_-/\lambda_- = -1/(k+1/2).
\end{aligned} \tag{B12}$$

One can further derive some useful relations between them:

$$\begin{aligned}
&\lambda_\pm = \lambda_\mp + 2\mu_\mp + 2, \qquad \mu_\pm = -(\mu_\mp + 2), \\
&\gamma_\pm = -\gamma_\mp, \qquad \mu_\pm\gamma_\pm + \mu_\mp\gamma_\mp = 2, \qquad \lambda_\pm + \mu_\pm = \lambda_\mp + \mu_\mp.
\end{aligned} \tag{B13}$$

Near the origin, the equations of motion behave asymptotically as

$$\varphi'' + \frac{2}{r}\varphi' - \frac{\lambda_\mp}{r^2}\varphi = 0, \tag{B14}$$

$$\begin{aligned}
\varphi_0^{*\prime\prime} + \frac{2}{r}\varphi_0^{*\prime} - \frac{\lambda_\pm}{r^2}\varphi_0^* = \omega&\left\{-(\varphi_1^{*\prime} + \frac{2}{r}\varphi_1^*) + \frac{\mu_\pm}{r}\varphi_2^* + \frac{\lambda_\mp+\mu_\mp}{r}\varphi_3^*\right\} \\
&+\eta\left\{\frac{1}{r}(2+\mu_\mp)\varphi_1^* - \mu_\pm\varphi_2^* - \frac{\lambda_\pm + 2\mu_\pm}{r}\varphi_2^* - \mu_\mp\left(\varphi_3^{*\prime} + \frac{1}{r}\varphi_3^*\right)\right\},
\end{aligned} \tag{B15}$$

$$\frac{\lambda_\pm}{r^2}\varphi_1^* = \frac{\mu_\pm}{r}(\varphi_2^{*\prime} + \frac{1}{r}\varphi_2^*) + \frac{\lambda_\mp+\mu_\mp}{r}(\varphi_3^{*\prime} + \frac{1}{r}\varphi_3^*) + \omega\varphi_0^{*\prime} - \eta\frac{\mu_\pm}{r}\varphi_0^*, \tag{B16}$$

$$
\begin{aligned}
&\varphi_2^{*\prime\prime} + \frac{2}{r}\varphi_2^{*\prime} - \frac{(\lambda_\pm+\mu_\pm)(1+\gamma_\pm)}{r^2}\varphi_2^* \\
&\qquad = \frac{\gamma_\pm}{r}\varphi_1^{*\prime} + \frac{\mu_\mp(1+\gamma_\pm)}{r^2}\varphi_3^* - \omega\frac{\gamma_\pm}{r}\varphi_0^* + \eta\left\{\gamma_\pm\varphi_0^{*\prime} - \frac{1}{r}\varphi_0^*\right\},
\end{aligned} \tag{B17}
$$

$$
\begin{aligned}
&\varphi_3^{*\prime\prime} + \frac{2}{r}\varphi_3^{*\prime} - \frac{\gamma_\mp\mu_\mp}{r^2}\varphi_3^* \\
&\qquad = \frac{(1+\gamma_\pm)}{r}\varphi_1^{*\prime} - \frac{\mu_\pm(1+\gamma_\pm)}{r^2}\varphi_2^* - \omega\frac{(1+\gamma_\mp)}{r}\varphi_0^* + \eta\gamma_\pm\varphi_0^{*\prime},
\end{aligned} \tag{B18}
$$

$$
\begin{aligned}
&\left\{\omega\varphi_0^* - (\varphi_1^{*\prime} + \frac{2}{r}\varphi_1^*) + \frac{\mu_\pm}{r}\varphi_2^* + \frac{(\lambda_\mp+\mu_\mp)}{r}\varphi_3^*\right\} \\
&\qquad = 3\eta\delta_2\varphi_0^* - \delta_2\left\{\frac{\mu_\pm}{r}\varphi_1^* - \left[\mu_\pm\varphi_2^{*\prime} + \frac{(\lambda_\mp-2)}{r}\varphi_2^*\right] + \mu_\mp\varphi_3^{*\prime}\right\} - \delta_1\left\{\varphi' - \frac{\mu_\mp}{r}\varphi\right\},
\end{aligned} \tag{B19}
$$

where we have used that, near the origin, the function $F(r)$ behaves as $F(r) = \pi + F'(0)r + \frac{1}{6}F'''(0)r^3 + \cdots$, which implies

$$
v(r) \sim \frac{1}{r} + O(r), \quad a_1(r) \sim -F'(0) + O(r^2), \quad a_2(r) \sim 2F'(0) + O(r^2). \tag{B20}
$$

In the expressions (B14-B19), the constants η, δ_1, and δ_2 are defined by

$$
\eta = g_Q F'(0), \quad \delta_1 = \frac{f_Q F'(0)}{2m_{\Phi^*}^2} = O(1/m_Q), \quad \delta_2 = \frac{[F'(0)]^2}{2m_{\Phi^*}^2} = O(1/m_Q^2). \tag{B21}
$$

With $F'(0) \sim 2ef_\pi$ and explicit heavy meson masses, one can evaluate the constants as $\delta_1 \sim 0.65,\ 0.29,\ 0.11$ and $\delta_2 \sim 0.31,\ 0.06,\ 0.01$ in case of $Q = s,\ c,\ b$, respectively. Note that, due to the vector potential $\mathbf{V}[\sim i(\hat{\mathbf{r}}\times\boldsymbol{\tau})/r$, near the origin] in the covariant derivative, the singular structure of the equations (B14-B19) is quite different from that of the free equations (4.1-4.6). Since these equations are not all independent, they yield only four independent asymptotic solutions that are finite at the origin. We list them in Table II, where c^i and c^i_α ($i = \mathrm{I}, \mathrm{II}, \mathrm{III}$, $\alpha = 0, 1, 2, 3$) for $\pi = (-1)^{k+1/2}$ states are constants satisfying

$$
\begin{aligned}
&(k+1/2)c_1^i - (k+3/2)c_2^i - (k+3/2)^2 c_3^i - (\omega-\eta)c_0^i = 0, \\
&(\omega - 3\eta\delta_2)c_0^i - [(k+5/2) + \delta_2(k-1/2)]c_1^i + (k-1/2)[1-2\delta_2(k+3/2)]c_2^i \\
&\qquad +(k+3/2)[(k-1/2) + \delta_2(k+5/2)]c_3^i + 2\delta_1(k+1)c^i = 0.
\end{aligned} \tag{B22}
$$

Since we have only two equations for five unknowns, we need to fix three of them. For example, we take ($c^{\mathrm{I}}_{1,2,3} = 0, c^{\mathrm{I}} = 1$), ($c^{\mathrm{II}} = c^{\mathrm{II}}_3 = 0, c^{\mathrm{II}}_2 = 1$), and ($c^{\mathrm{III}} = c^{\mathrm{III}}_2 = 0, c^{\mathrm{III}}_3 = 1$), which yield three solution sets given in Table II.

On the other hand, at large r where heavy mesons become free from the interactions with the soliton, the equations of motion (B1-B6) approach asymptotically to Eqs. (4.1-4.6). Since we are interested in the bound states with $\omega^2 < m_\Phi^2, m_{\Phi^*}^2$, instead of the spherical Bessel function $j_\ell(x)$ in the free solutions, the asymptotic solutions are written in terms of the modified spherical Hankel functions $\tilde{k}_\ell(x)$ satisfying

$$
\tilde{k}_\ell'' + \frac{2}{x}\tilde{k}_\ell' - \left[1 + \frac{\ell(\ell+1)}{x^2}\right]\tilde{k}_\ell = 0, \tag{B23}
$$

with recurrence relations

$$
\begin{aligned}
&\tilde{k}_{\ell-1}(x) - \tilde{k}_{\ell+1}(x) = [(2\ell+1)/x]\tilde{k}_{\ell}(x), \\
&\ell\tilde{k}_{\ell-1}(x) + (\ell+1)\tilde{k}_{\ell+1}(x) = (2\ell+1)\tilde{k}'_{\ell}(x).
\end{aligned}
\tag{B24}
$$

Listed in Table III are four independent asymptotic solutions which decay exponentially at infinity, where $Q = \sqrt{m_{\Phi}^2 - \omega^2}$ and $Q^* = \sqrt{m_{\Phi^*}^2 - \omega^2}$.

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[21] The most essential point of Ansatz II is that the vector mesons are much heavier than the pseudoscalar mesons.

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FIGURES

FIG. 1. Heavy meson masses and the mass formula (2.13). The pseudoscalar meson mass m_Φ and the vector meson mass m_{Φ^*} obtained using Eq. (2.13) are given by solid and dashed lines, respectively.

FIG. 2. Radial wavefunctions of the heavy meson bound states in arbitrary scale. Solid lines are those obtained by solving the equations exactly, while dashed lines and dash-dotted lines are the approximate solutions of Ans$\ddot{}a$tze I and II, respectively.

FIG. 3. Binding energies, $\omega - m_\Phi$, of $k^\pi{=}\frac{1}{2}^+$ ground state, $\frac{1}{2}^+$ excited state, and $\frac{1}{2}^-$ state. Solid lines represent the exact solutions and dashed (dash-dotted) lines correspond to Ansatz I (II).

FIG. 4. Binding energies, $\omega - m_\Phi$, of the bound states with k^π as functions of the heavy meson mass. Solid (dashed) lines denote the positive (negative) parity states.

FIG. 5. Radial wavefunctions of the antiflavored heavy meson bound states in arbitrary scale.

FIG. 6. Binding energies, $|\omega| - m_\Phi$, of the bound states as functions of the antiflavored heavy meson mass. Solid (dashed) lines denote the positive (negative) parity states.

TABLES

TABLE I. Eight grand spin eigenstates.

	$\pi = (-1)^{k+1/2}$				$\pi = (-1)^{k-1/2}$		
state	s	ℓ	j	state	s	ℓ	j
$\mathcal{Y}^{(+)}_{k\,m_k}$	0	j	$k-1/2$	$\mathcal{Y}^{(-)}_{k\,m_k}$	0	j	$k+1/2$
$\boldsymbol{\mathcal{Y}}^{(1)}_{k\,m_k}$	1	j	$k-1/2$	$\boldsymbol{\mathcal{Y}}^{(4)}_{k\,m_k}$	1	j	$k+1/2$
$\boldsymbol{\mathcal{Y}}^{(2)}_{k\,m_k}$	1	$j-1$	$k+1/2$	$\boldsymbol{\mathcal{Y}}^{(5)}_{k\,m_k}$	1	$j-1$	$k-1/2$
$\boldsymbol{\mathcal{Y}}^{(3)}_{k\,m_k}$	1	$j+1$	$k+1/2$	$\boldsymbol{\mathcal{Y}}^{(6)}_{k\,m_k}$	1	$j+1$	$k-1/2$

TABLE II. Four independent asymptotic solutions near the origin.

parity	$\pi = (-1)^{k+1/2}$	$\pi = (-1)^{k-1/2}$
Solution I	$\varphi \sim r^{k+1/2} + O(r^{k+5/2})$ $\varphi_0^* \sim c_0^{\mathrm{I}} r^{k-1/2} + O(r^{k+3/2})$ $\varphi_{1,2,3}^* \sim O(r^{k+5/2})$	$\varphi \sim r^{k-1/2} + O(r^{k+3/2})$ $\varphi_0^* \sim O(r^{k+1/2})$ $\varphi_{1,2,3}^* \sim O(r^{k+3/2})$
Solution II	$\varphi, \varphi_3^* \sim O(r^{k+5/2})$ $\varphi_0^* \sim c_0^{\mathrm{II}} r^{k-1/2} + O(r^{k+3/2})$ $\varphi_1^* \sim c_1^{\mathrm{II}} r^{k+1/2} + O(r^{k+5/2})$ $\varphi_2^* \sim r^{k+1/2} + O(r^{k+5/2})$	$\varphi, \varphi_3^* \sim O(r^{k+3/2})$ $\varphi_0^* \sim O(r^{k+1/2})$ $\varphi_1^* \sim +r^{k-1/2} + O(r^{k+3/2})$ $\varphi_2^* \sim -r^{k-1/2} + O(r^{k+3/2})$
Solution III	$\varphi, \varphi_2^* \sim O(r^{k+5/2})$ $\varphi_0^* \sim c_0^{\mathrm{III}} r^{k-1/2} + O(r^{k+3/2})$ $\varphi_1^* \sim c_1^{\mathrm{III}} r^{k+1/2} + O(r^{k+5/2})$ $\varphi_3^* \sim r^{k+1/2} + O(r^{k+5/2})$	$\varphi, \varphi_2^* \sim O(r^{k+3/2})$ $\varphi_0^* \sim O(r^{k+1/2})$ $\varphi_1^* \sim (k-1/2) r^{k-1/2} + O(r^{k+3/2})$ $\varphi_3^* \sim r^{k-1/2} + O(r^{k+3/2})$
Solution IV	$\varphi \sim O(r^{k+1/2})$ $\varphi_0^* \sim r^{k-1/2} + O(r^{k+3/2})$ $\varphi_1^* \sim \lambda_+ r^{k-3/2} + O(r^{k+1/2})$ $\varphi_2^* \sim r^{k-3/2} + O(r^{k+1/2})$ $\varphi_3^* \sim \mu_+ r^{k-3/2} + O(r^{k+1/2})$	$\varphi \sim O(r^{k+3/2})$ $\varphi_0^* \sim r^{k+1/2} + O(r^{k+5/2})$ $\varphi_1^* \sim O(r^{k+3/2})$ $\varphi_2^* \sim O(r^{k+3/2})$ $\varphi_3^* \sim O(r^{k+3/2})$

TABLE III. Four independent asymptotic solutions at large r.

parity	$\pi = (-1)^{k+1/2}$	$\pi = (-1)^{k-1/2}$
Solution I	$\varphi \sim \tilde{k}_{k-1/2}(Qr)$ $\varphi^*_{0,1,2,3} \sim 0$	$\varphi \sim \tilde{k}_{k+1/2}(Qr)$ $\varphi^*_{0,1,2,3} \sim 0$
Solution II	$\varphi^*_2 \sim \tilde{k}_{k-1/2}(Q^*r)$ $\varphi, \varphi^*_{0,1,3} \sim 0$	$\varphi^*_2 \sim \tilde{k}_{k+1/2}(Q^*r)$ $\varphi, \varphi^*_{0,1,3} \sim 0$
Solution III	$\varphi, \varphi^*_2 \sim 0$ $\varphi^*_0 \sim (k+1/2)(Q^*/\omega)\tilde{k}_{k+1/2}(Q^*r)$ $\varphi^*_1 \sim (k+1/2)\tilde{k}_{k-1/2}(Q^*r)$ $\varphi^*_3 \sim \tilde{k}_{k-1/2}(Q^*r)$	$\varphi, \varphi^*_2 \sim 0$ $\varphi^*_0 \sim -(k+1/2)(Q^*/\omega)\tilde{k}_{k-1/2}(Q^*r)$ $\varphi^*_1 \sim (k+1/2)\tilde{k}_{k+1/2}(Q^*r)$ $\varphi^*_3 \sim -\tilde{k}_{k+1/2}(Q^*r)$
Solution IV	$\varphi, \varphi^*_2 \sim 0$ $\varphi^*_0 \sim (k+3/2)(Q^*/\omega)\tilde{k}_{k+1/2}(Q^*r)$ $\varphi^*_1 \sim (k+3/2)\tilde{k}_{k+3/2}(Q^*r)$ $\varphi^*_3 \sim -k_{k+3/2}\tilde{k}_{k+3/2}(Q^*r)$	$\varphi, \varphi^*_2 \sim 0$ $\varphi^*_0 \sim -(k-1/2)(Q^*/\omega)\tilde{k}_{k-1/2}(Q^*r)$ $\varphi^*_1 \sim (k-1/2)\tilde{k}_{k-3/2}(Q^*r)$ $\varphi^*_3 \sim \tilde{k}_{k-3/2}(Q^*r)$